

Degree bounds for the toric ideal of a matroid

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ABSTRACT. Describing minimal generating set of a toric ideal, or the minimum degree in which it is generated, is a well-studied and difficult problem. In 1980 White conjectured that the toric ideal I_M of a matroid M is generated by quadratic binomials corresponding to symmetric exchanges.

The weak version of White's conjecture asserts that the ideal I_M is generated in degree 2. We give the first bound on degree that depends only on the rank of a matroid. Namely, we prove that the toric ideal of a matroid of rank r is generated in degree at most $(r+3)!$. As a corollary we obtain that checking if White's conjecture is true for matroids of a fixed rank is decidable.

In [11] we proved White's conjecture 'up to saturation'. That is, that for every matroid M homogeneous parts of the ideal J_M generated by quadratic binomials corresponding to symmetric exchanges and of the toric ideal I_M are equal starting from some degree. We extend this result by proving that there exists a function f such that for every matroid M of rank r homogeneous parts of the ideals J_M and I_M are equal starting from degree $f(r)$.

Additionally, we observe that White's conjecture implies that for every matroid whose ground set can be partitioned into disjoint bases, the basis graph restricted to bases appearing in some partition is connected. We provide sufficient conditions for the opposite implication.

1. Introduction

Let M be a matroid on a ground set E with the set of bases \mathfrak{B} and the rank function $r : \mathcal{P}(E) \rightarrow \mathbb{N}$. The rank of M , that is $r(E)$, we denote simply by r .

For a fixed field \mathbb{K} consider a \mathbb{K} -homomorphism φ_M between polynomial rings:

$$\varphi_M : \mathbb{K}[y_B : B \in \mathfrak{B}] \ni y_B \rightarrow \prod_{e \in B} x_e \in \mathbb{K}[x_e : e \in E].$$

The *toric ideal of a matroid* M , denoted by I_M , is the kernel of the map φ_M . For a representable matroid M the toric variety associated with the toric ideal I_M has a very nice embedding as a subvariety of a Grassmannian [8]. It is the closure of the torus orbit of the point of the Grassmannian corresponding to the matroid M .

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Neil White in 1980 made three conjectures of growing difficulty that describe generators of the ideal I_M .

CONJECTURE 1 (White, [21]). *The toric ideal of a matroid is generated in degree 2.*

The family \mathfrak{B} of bases of M satisfies *symmetric exchange property* (the reader is referred to [16] for background of matroid theory, and to [12] for other exchange properties). That is, for every bases B_1, B_2 and $e \in B_1 \setminus B_2$ there exists $f \in B_2 \setminus B_1$, such that both sets $B'_1 = (B_1 \setminus e) \cup f$ and $B'_2 = (B_2 \setminus f) \cup e$ are bases. In this case we say that the quadratic binomial $y_{B_1}y_{B_2} - y_{B'_1}y_{B'_2}$ corresponds to *symmetric exchange*. It is clear that such binomials belong to the ideal I_M .

CONJECTURE 2 (White, [21]). *The toric ideal of a matroid is generated by quadratic binomials corresponding to symmetric exchanges.*

CONJECTURE 3 (White, [21]). *The toric ideal of a matroid considered in the noncommutative polynomial ring $\mathbb{K}\langle y_B : B \in \mathfrak{B} \rangle$ is generated by quadratic binomials corresponding to symmetric exchanges.*

Conjecture 3, the strongest among White's conjectures describing generators of the ideal I_M , turned out to be equivalent to Conjecture 2 (see the discussion in Section 4 of [11]).

Since every toric ideal is generated by binomials, it is not hard to rephrase the above conjectures in the combinatorial language. Conjecture 1 asserts that if two multisets of bases of a matroid have equal union (as a multiset), then one can pass between them by a sequence of steps, in each step exchanging two bases for another two bases of the same union (as a multiset). In Conjecture 2 additionally each step corresponds to a symmetric exchange. In Conjecture 3 we take sequences of bases instead of multisets, and similarly each step corresponds to a symmetric exchange. Actually, this is the original formulation due to White. We immediately see that the conjectures do not depend on the field \mathbb{K} .

White's conjectures are known to be true for many special classes of matroids: graphic matroids [1], strongly base orderable matroids [11] (so also for transversal matroids), sparse paving matroids [3], and for matroids of rank at most 3 [10] (see also other related papers [2, 4, 9, 17, 18]).

The first result valid for arbitrary matroids was confirmation of White's Conjecture 2 'up to saturation'. Let \mathfrak{m} be the ideal generated by all variables in the polynomial ring $\mathbb{K}[y_B : B \in \mathfrak{B}]$ (so-called *irrelevant ideal*). Recall that the ideal $I : \mathfrak{m}^\infty = \{a \in S_M : a\mathfrak{m}^n \subset I \text{ for some } n \in \mathbb{N}\}$ is called the *saturation* of an ideal I with respect to the ideal \mathfrak{m} . Notice that the ideal I_M , as a prime ideal, is saturated. Let J_M be the ideal generated by quadratic binomials corresponding to symmetric exchanges. Clearly, $J_M \subset I_M$ and Conjecture 2 asserts that the ideals J_M and I_M are equal. In [11] we prove that their saturations with respect to \mathfrak{m} are equal.

Recall that two homogeneous ideals have equal saturations with respect to the ideal generated by all variables if and only if their homogeneous parts are equal starting from some degree. Thus we can rephrase the above in a following way.

THEOREM 4 (Lasoń, Michałek, [11]). *Homogeneous parts of the toric ideal of a matroid and of the ideal generated by quadratic binomials corresponding to symmetric exchanges are equal starting from some degree.*

Here we make another step towards White's conjecture. We bound degree in which the toric ideal I_M of a matroid M is generated, and we bound degree starting from which homogeneous parts of the ideals J_M and I_M are equal.

By Hilbert's basis theorem the ideal I_M is finitely generated. However, it is not easy to give any explicit bound on degree in which it is generated. A bound follows from a more general theorem about toric ideals. Theorem 13.14 from [19] asserts that if a graded set $\mathcal{A} \subset \mathbb{Z}^d$ generates a normal semigroup, then the corresponding toric ideal $I_{\mathcal{A}}$ is generated in degree at most d . For a matroid M we consider the set $\mathcal{A} = \{\chi_B : B \in \mathfrak{B}\} \subset \mathbb{Z}^{|E|}$, where χ_B is a characteristic function of B in E . By Theorem 1 from [20] \mathcal{A} generates a normal semigroup (it is also an easy consequence of the matroid union theorem). The toric ideal corresponding to \mathcal{A} is the ideal I_M . Hence, the toric ideal of a matroid is generated in degree at most the size of its ground set.

If we fix the size of the ground set, then there are only finitely many matroids on it. So a common bound is not surprising. When we fix only the rank, then the number of matroids of that rank is infinite. We prove that in this case there is also a common bound on degree.

THEOREM 5. *The toric ideal associated to a matroid of rank r is generated in degree at most $(r+3)!$.*

We get a method of checking White's conjectures for matroids of a fixed rank.

COROLLARY 6. *Checking if Conjecture 1, 2 or 3 is true for matroids of a fixed rank is decidable (it is enough to check connectivity of a finite number of graphs).*

We get a new class of discrete polymatroids for which White's Conjecture 1 is true (for the extension of White's conjectures to discrete polymatroids see [9]).

COROLLARY 7. *Let P be a discrete polymatroid which is a join of $c \geq \frac{1}{2}(r+3)!$ copies of a matroid M of rank r (a basis of P is a union, as a multiset, of c bases of M). Then the toric ideal I_P is generated in degree 2.*

We extend Theorem 4. A careful reading of its proof gives that homogeneous parts of the ideals J_M and I_M are equal starting from degree around $r|E||\mathfrak{B}|$. We prove a bound that depends only on the rank of a matroid.

THEOREM 8. *There exists a function f such that homogeneous parts of the toric ideal of a matroid of rank r and of the ideal generated by quadratic binomials corresponding to symmetric exchanges are equal starting from degree $f(r)$.*

In particular, this allows us to generate by quadratic binomials corresponding to symmetric exchanges binomials $y_{B_1} \cdots y_{B_k} - y_{B'_1} \cdots y_{B'_k} \in I_M$ of large degree with respect to the rank ($k \gg r$) for which bases B_1, \dots, B_k are pairwise disjoint. In this case Theorem 4 can not be applied.

The next Section 2 contains preliminaries, in particular language used later. We discuss there how White's conjectures translate into problems on graphs on bases of a matroid. We show that White's Conjecture 2 implies that for every matroid whose ground set can be partitioned into disjoint bases, the basis graph restricted to bases appearing in some partition is connected. We provide sufficient conditions for the opposite implication. In the last Section 3 we prove Theorems 5, 8 and Corollaries 6, 7. By discussion after Remark 15 from [11] we get the following.

REMARK 9. *Theorems 5, 8, Corollaries 6, 7 are true for discrete polymatroids.*

2. Graphs on bases of a matroid

We say that two bases of a matroid are *neighbouring* if one is obtained from another by a symmetric exchange. That is, if their symmetric difference has two elements. A graph on bases of a matroid M with edges between neighbouring bases is called the *basis graph* of a matroid, and denoted by $\mathfrak{B}(M)$. Basis graphs have been studied in 1960s and 1970s, and they are well understood. In particular, basis graphs are Hamiltonian (with two trivial exceptions), even a characterization is known (see [13, 14] and references within).

For $k \geq 1$, a k -matroid is a matroid whose ground set can be partitioned into k pairwise disjoint bases. We call a basis of a k -matroid *complementary* if its complement can be partitioned into $k - 1$ pairwise disjoint bases. That is, when it is an element of some partition of the ground set into bases. When \mathfrak{B} is the set of bases of a k -matroid, then the set of complementary bases we denote by \mathfrak{B}^c .

Blasiak [1] proposed a very nice and simple translation of the problem of generating the ideal I_M to the problem of connectivity of some graphs naturally associated to k -matroids. We are going to use this approach for the proof of Theorem 5 and Corollary 6. For $k \geq 3$, the k -base graph of a k -matroid M , denoted by $\mathfrak{B}_k(M)$, is a graph on sets of k pairwise disjoint bases of M (partitions of the ground set into bases), where edges join vertices with nonempty intersection. That is, sets of bases $\{B_1, \dots, B_k\}$ and $\{B'_1, \dots, B'_k\}$ are connected in $\mathfrak{B}_k(M)$ if for some i, j equality $B_i = B'_j$ holds. A simple corollary of the proof of Proposition 2.1 from [1] gives the following.

PROPOSITION 10 (Blasiak, [1]). *Let \mathfrak{C} be a class of matroids that is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Then the following conditions are equivalent:*

- (1) *for every $k > n$ and for every k -matroid M in \mathfrak{C} the k -base graph $\mathfrak{B}_k(M)$ is connected,*
- (2) *for every matroid M in \mathfrak{C} the ideal I_M is generated in degree at most n .*

In particular, in order to prove Conjecture 1 it is enough to show that for every $k > 2$ and for every k -matroid M the k -base graph $\mathfrak{B}_k(M)$ is connected.

Here we propose another approach to White's conjecture. Consider other graphs that can be naturally associated to k -matroids. The *complementary basis graph* of a k -matroid, denoted by $\mathfrak{B}^c(M)$, is a graph on complementary bases of M with edges between neighbouring bases. That is, the complementary basis graph of a k -matroid is the restriction of its basis graph to complementary bases $\mathfrak{B}^c(M) = \mathfrak{B}(M)|_{\mathfrak{B}^c}$.

Graphs $\mathfrak{B}^c(M)$ have been already studied for 2-matroids. In 1985 Farber, Richter and Shank [7] proved that for a graphic 2-matroid M the graph $\mathfrak{B}^c(M)$ is connected, they also conjectured connectivity for arbitrary 2-matroids. In [1] after the proof of Proposition 2.1 Blasiak observes the following easy equivalence.

PROPOSITION 11 (Blasiak, [1]). *Let \mathfrak{C} be a class of matroids that is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Then the following conditions are equivalent:*

- (1) *for every 2-matroid M in \mathfrak{C} the complementary basis graph $\mathfrak{B}^c(M)$ is connected,*

- (2) for every matroid M in \mathfrak{C} , elements of degree 2 in I_M considered in the noncommutative polynomial ring $\mathbb{K}\langle y_B : B \in \mathfrak{B} \rangle$ are generated by quadratic binomials corresponding to symmetric exchanges.

We state the following two conjectures.

CONJECTURE 12. *Complementary basis graph of a k -matroid is connected.*

CONJECTURE 13. *Let M be a matroid of rank r on the ground set E of size $kr + 1$, for $k \geq 2$. Suppose $x, y \in E$ are two elements such that both sets $E \setminus x$ and $E \setminus y$ can be partitioned into k pairwise disjoint bases. Then there exist partitions of $E \setminus x$ and $E \setminus y$ into k pairwise disjoint bases which share a common basis.*

As we show there is a strong relation between the above conjectures and the strongest White's Conjecture 3.

PROPOSITION 14. *Let \mathfrak{C} be a class of matroids that is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Then, considered for all matroids in \mathfrak{C} , the following implications between conjectures hold:*

- (1) *the strongest White's Conjecture 3 implies complementary basis graph Conjecture 12,*
- (2) *conjunction of complementary basis graph Conjecture 12 and Conjecture 13 implies the strongest White's Conjecture 3.*

PROOF. We begin with the implication (1). Let M be a k -matroid in \mathfrak{C} , and let B_1, B'_1 be complementary bases in M . So, there exist bases $B_2, \dots, B_k, B'_2, \dots, B'_k$ such that entries of the sequences $\mathcal{A} = (B_1, \dots, B_k)$ and $\mathcal{A}' = (B'_1, \dots, B'_k)$ form partitions of the ground set E . Then $b = y_{B_1} \cdots y_{B_k} - y_{B'_1} \cdots y_{B'_k} \in I_M$, or equivalently sequences of bases \mathcal{A} and \mathcal{A}' have equal union (as a multiset). By the assumption, we can generate b using quadratic binomials corresponding to symmetric exchanges, or equivalently we can pass between \mathcal{A} and \mathcal{A}' by a sequence of steps, in each step making a symmetric exchange. Notice that all bases appearing during this process are complementary bases in M . Observe that the first bases of two sequences joined by a step are either the same or neighbouring. Thus we get a path in $\mathfrak{B}^c(M)$ between B_1 and B'_1 .

For the implication (2), by Propositions 10 and 11 it is enough to show that for every $k \geq 3$ and for every k -matroid M in \mathfrak{C} the k -base graph $\mathfrak{B}_k(M)$ is connected, and for every 2-matroid M in \mathfrak{C} the complementary basis graph $\mathfrak{B}^c(M)$ is connected. The second part we get directly from complementary basis graph Conjecture 12. Let M be a k -matroid in \mathfrak{C} (for $k \geq 3$), and let $\{B_1, \dots, B_k\}, \{B'_1, \dots, B'_k\}$ be two vertices in $\mathfrak{B}_k(M)$. From the assumption, complementary bases B_1, B'_1 are joined by a path in $\mathfrak{B}^c(M)$. Each vertex of the path is a complementary basis, so for each there exist bases completing the partition of the ground set. Thus, it is enough to show that vertices of $\mathfrak{B}_k(M)$ corresponding to neighbouring bases are connected in $\mathfrak{B}_k(M)$. If $B_1 \triangle B'_1 = \{x, y\}$, then consider matroid $M|_{E \setminus B_1 \cap B'_1}$. It satisfies assumptions of Conjecture 13 with points x, y . Thus in $\mathfrak{B}_k(M)$ there are vertices $\{B_1, B'_2, \dots, B'_k\}, \{B'_1, B_2, \dots, B_k\}$ connected by an edge. The first one is connected by an edge with $\{B_1, \dots, B_k\}$, while the second with $\{B'_1, \dots, B'_k\}$. We get that any two vertices $\{B_1, \dots, B_k\}, \{B'_1, \dots, B'_k\}$ in $\mathfrak{B}_k(M)$ are connected. \square

PROPOSITION 15. *If $k \geq 2^{r-1} + 1$, then Conjecture 13 holds.*

PROOF. Proof by contradiction. The set $E \setminus y$ can be partitioned into k pairwise disjoint bases, let B_1, \dots, B_k be such a partition. Without loss of generality $x \in B_1$. If the assertion is not true, then for each $i = 2, \dots, k$ basis B_i can not be completed to a partition of $E \setminus x$ into k bases. Thus from the matroid union theorem follows that for each $i = 2, \dots, k$ there is a set $A_i \subset E \setminus (x \cup B_i)$ such that $(k-1)r(A) < |A|$. On the other hand, since $E \setminus (y \cup B_i)$ has a partition into $k-1$ pairwise disjoint bases, for every set $A \subset E \setminus (y \cup B_i)$ inequality $(k-1)r(A) \geq |A|$ holds. Thus for each i we have $y \in A_i$, $r(A_i) = r(A_i \setminus y)$, and $(k-1)r(A_i \setminus y) = |A_i \setminus y|$. The last equality implies that for every basis B_j (for $j \neq i$) $|B_j \cap A_i| = r(A_i)$. Moreover, since there is equality in the inequality $(k-1)r(A_i \setminus y) \geq |A_i \setminus y|$, each A_i is closed, and it is equal to the closure of $B_j \cap A_i$ in $E \setminus (x \cup B_i)$. Consider the sets $B_1 \cap A_i \subset B_1 \setminus x$. None of them is empty, since otherwise $r(A_i) = 0$ and y would be a loop. Thus, since there are $k-1 \geq 2^{r-1}$ of them, for some $i \neq j$ equality $B_1 \cap A_i = B_1 \cap A_j$ holds. But since $A_i = \overline{B_1 \cap A_i}^{(E \setminus (x \cup B_i))}$ and $A_j = \overline{B_1 \cap A_j}^{(E \setminus (x \cup B_j))}$ we get that the set $A := A_i \cup A_j$ is closed and $|B_l \cap A| = r(A)$ for every $l = 1, \dots, k$. Therefore, $|A| = kr(A) + 1$, which contradicts the assumption that $E \setminus x$ can be partitioned into k pairwise disjoint bases. \square

3. Main results

In order to prove Theorem 5 we need to prove a slightly more general statement.

THEOREM 16. *For every integers $r \geq 1, s \geq 0, k \geq s(r+1)! + (r+3)!$ and for every*

- (1) *k -matroid M of rank r with k pairwise disjoint bases B_1, \dots, B_k , and*
- (2) *its restriction N to $k-s$ pairwise disjoint bases D_{s+1}, \dots, D_k*

there exist

- (3) *a vertex in $\mathfrak{B}_k(M)$ connected by an edge* with $\{B_1, \dots, B_k\}$, and*
- (4) *a vertex in $\mathfrak{B}_{k-s}(N)$ connected by an edge* with $\{D_{s+1}, \dots, D_k\}$*

*which have nonempty intersection – a common basis. (*this can be a loop)*

PROOF. The proof goes by induction on r . When $r = 1$, then the statement becomes trivial.

Suppose $r \geq 2$. Since $k \geq r+1$, then without loss of generality we can assume that B_1 and D_{s+1} do not intersect. If there is a vertex in $\mathfrak{B}_k(M)$ containing both B_1 and D_{s+1} , then the assertion is true. Suppose contrary – there is no such vertex. Hence, due to matroid union theorem (firstly formulated for graphs in [15]) there exists a set $A \subset E \setminus (B_1 \cup D_{s+1})$ such that

$$(k-2)r(A) < |A|.$$

Of course $r(A) > 0$, since otherwise A would have to be empty (in a k -matroid M of rank $r > 0$ there are no loops) and we would have $0 < 0$. We know also that $r > r(A)$, since otherwise we would have $(k-2)r < |A| \leq |E \setminus (B_1 \cup D_{s+1})| = (k-2)r$.

Let $A_i = B_i \cap A$ for $i = 2, \dots, k$. We have $|A_i| \leq r(A)$, since each B_i is a basis. We get that

$$(k-2)r(A) < |A| = |A_2| + \dots + |A_k| \leq (k-1)r(A).$$

Thus for all $i = 2, \dots, k$, except at most $r(A) - 1$, the equality $|A_i| = r(A)$ holds. Without loss of generality it holds for $i = r+1, \dots, k$.

Let $A' = A_{r+1} \cup \dots \cup A_k$ and let $E' = B_{r+1} \cup \dots \cup B_k$. We are going to reduce our problem to the $(k-r)$ -matroid $M' = M|_{E'}$ (restriction of M to the set E'), and use the set A' to split the problem for M' into smaller instances – for $M'|_{A'}$ (restriction of M' to A') and for M'/A' (contraction of A' in M').

Again without loss of generality we can assume that D_{s+1}, \dots, D_{s+t} are the only bases among D_{s+1}, \dots, D_k that have nonempty intersection with some of the bases B_1, \dots, B_r . That is, bases D_{s+t+1}, \dots, D_k are contained in the set E' . Moreover, $t \leq r^2$ since each B_i can be intersected by at most r bases D_j .

Let $C_j = D_j \cap A'$ for $j = s+r^2+1, \dots, k$. We have $|C_j| \leq r(A') = r(A)$, since each D_j is a basis. In order to reduce the problem for $M'|_{A'}$ and M'/A' we need bases D_j satisfying $|C_j| = r(A)$. Since D_{s+r^2+1}, \dots, D_k cover all except $(s+r^2)r$ points of E we get

$$(k - (s+r^2))r(A) - (s+r^2)r \leq (k-r^2)r(A) - (s+r^2)r = |A'| - (s+r^2)r \leq$$

$$\leq |(D_{s+r^2+1} \cup \dots \cup D_k) \cap A'| = |C_{s+r^2+1}| + \dots + |C_k| \leq (k - (s+r^2))r(A).$$

Thus for all $i = s+r^2+1, \dots, k$, except at most $(s+r^2)r$, the equality $|C_i| = r(A)$ holds. Without loss of generality it holds for $i = (s+r^2)(r+1)+1, \dots, k$. Denote $s' = (s+r^2)(r+1) - r$. Now we can pass to matroids $M'|_{A'}$ and M'/A' . We have

- (1) $(k-r)$ -matroid $M'|_{A'}$ with $k-r$ pairwise disjoint bases $B_{r+1} \cap A', \dots, B_k \cap A'$, and
- (2) its restriction $N'|_{A'}$ to $k-r-s'$ pairwise disjoint bases $D_{s'+r+1} \cap A', \dots, D_k \cap A'$, and
- (1) $(k-r)$ -matroid M'/A' with $k-r$ pairwise disjoint bases $B_{r+1} \setminus A', \dots, B_k \setminus A'$, and
- (2) its restriction N'/A' to $k-r-s'$ pairwise disjoint bases $D_{s'+r+1} \setminus A', \dots, D_k \setminus A'$.

In both cases we use inductive hypothesis. It is elementary to check that the inequalities from the assumption of the theorem are satisfied (remember $r > r(A') > 0$):

$$k-r \geq s(r+1)! + (r+3)! - r \geq s'(r(A')+1)! + (r(A')+3)!,$$

$$k-r \geq s(r+1)! + (r+3)! - r \geq s'(r-r(A')+1)! + (r-r(A')+3)!.$$

By the assertion of the theorem we get that there exists

- (3) a vertex in $\mathfrak{B}_{k-r}(M \cap A')$ connected by an edge* with $\{B_{r+1} \cap A', \dots, B_k \cap A'\}$, and
- (4) a vertex in $\mathfrak{B}_{k-r-s'}(N \cap A')$ connected by an edge* with $\{D_{s'+r+1} \cap A', \dots, D_k \cap A'\}$

which have nonempty intersection:

$$\{B_{r+1} \cap A', \dots, B_k \cap A'\} - \{G, T_i\} - \{G, V_j\} - \{D_{s'+r+1} \cap A', \dots, D_k \cap A'\},$$

where ‘ $-$ ’ denotes an edge* in the relevant graph. We get also that there exists

- (3) a vertex in $\mathfrak{B}_{k-r}(M/A')$ connected by an edge* with $\{B_{r+1} \setminus A', \dots, B_k \setminus A'\}$, and
- (4) a vertex in $\mathfrak{B}_{k-r-s'}(N'/A')$ connected by an edge* with $\{D_{s'+r+1} \setminus A', \dots, D_k \setminus A'\}$

which have nonempty intersection:

$$\{B_{r+1} \setminus A', \dots, B_k \setminus A'\} - \{H, U_i\} - \{H, W_j\} - \{D_{s'+r+1} \setminus A', \dots, D_k \setminus A'\}.$$

Using the fact that the union of any basis of $M'|_{A'}$ and any basis of M'/A' gives a basis of M' (and for N' analogously) we can now join everything together:

$$\begin{aligned} & \{B_1, \dots, B_k\} - \{B_1, \dots, B_r, G \cup H, T_i \cup U_i\} - \\ & - \{D_{s+1}, \dots, D_{s'+r}, G \cup H, V_j \cup W_j\} - \{D_{s+1}, \dots, D_k\}. \end{aligned}$$

This proves the assertion. \square

PROOF OF THEOREM 5. We apply Proposition 10 to the class of matroids of rank r , and $n = (r+3)!$. Clearly, this class is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Theorem 16 applied to r and $s = 0$ asserts that for every $k \geq (r+3)!$ the k -base graph $\mathfrak{B}_k(M)$ of a k -matroid M of rank r is connected. We get even that in these graphs every two vertices are connected by a path of length at most 3. This completes the proof of Theorem 5. \square

PROOF OF COROLLARY 6. Again using Proposition 10, in order to check if Conjecture 1 is true for matroids of rank r it is enough to check if for every k from the range $(r+3)! \geq k > 2$ and for every k -matroid M of rank r the k -base graph $\mathfrak{B}_k(M)$ is connected. That is, to check connectivity of a finite number of graphs. To check if Conjectures 1 and 3 are equivalent for matroids of rank r it also suffices to check connectivity of a finite number of graphs, see Proposition 11. This completes the proof of Corollary 6. \square

PROOF OF COROLLARY 7. We will prove the following claim. Let P be a discrete polymatroid which is a join of d copies of a matroid M . Suppose that the toric ideal I_M is generated in degree $2d$. Then the toric ideal I_P is generated in degree 2.

Let $D_1, \dots, D_k, D'_1, \dots, D'_k$ be bases of P with $y_{D_1} \cdots y_{D_k} - y_{D'_1} \cdots y_{D'_k} \in I_P$. For $i = 1, \dots, k$ and $j = 1, \dots, c$ let B_i^j and $B'_i{}^j$ be bases of M such that $D_i = \bigcup_j B_i^j$ and $D'_i = \bigcup_j B'_i{}^j$. Then $\prod_{i,j} y_{B_i^j} - \prod_{i,j} y_{B'_i{}^j} \in I_M$.

Observe that when one exchanges bases B_i^j and $B'_i{}^j$ between bases D_i and D'_i of P , then the corresponding elements of I_P differ by an element generated in degree 2. Thus we can always rearrange bases B_i^j (and $B'_i{}^j$) into an arbitrary k multisets of c bases. Since I_M is generated in degree $2d$, one can pass between the multisets of bases $\{B_i^j : i, j\}$ and $\{B'_i{}^j : i, j\}$ by a sequence of steps, in each step exchanging $2d$ bases for another $2d$ bases of the same union (as a multiset). We partition these $2d$ bases into an arbitrary 2 parts of d bases. Each part corresponds to a basis of P . This way we are able to pass between the multisets of bases $\{D_i : i\}$ and $\{D'_i : i\}$ of P by a sequence of steps, in each step exchanging 2 bases for another 2 bases of the same union (as a multiset). \square

A matroid obtained from a matroid M by replacing every element by k parallel elements we call the k -th blow up of the matroid M . In order to prove Theorem 8 we need the following structural lemma, which roughly says that if a matroid contains a large enough number of disjoint bases, then one can find inside it a submatroid which is isomorphic to the k -th blow up of a basis.

LEMMA 17. *For every rank $r \geq 1$ and for every integer $k \geq 1$ there exist an integer $n = n(r, k) \geq 1$, such that if M is an n -matroid of rank r with disjoint bases B_1, \dots, B_n , then there are disjoint bases B'_1, \dots, B'_n such that:*

- (1) B'_1, \dots, B'_n are obtained from B_1, \dots, B_n by symmetric exchanges, that is $y_{B_1} \cdots y_{B_n} - y_{B'_1} \cdots y_{B'_n} \in J_M$,
- (2) among them there are k bases $B'_{i_1}, \dots, B'_{i_k}$, such that M restricted to their union contains a submatroid isomorphic to the k -th blow up of a basis in which $B'_{i_1}, \dots, B'_{i_k}$ are bases. That is, one can label elements of their union with labels $1, \dots, r$, each B'_{i_j} with distinct labels, such that every set of r elements of distinct labels forms a basis in M .

PROOF. The proof goes by induction on the rank r . If $r = 1$ then M itself is an n -th blow up of a basis. Thus $n(1, k) = k$.

Suppose $r \geq 2$. Let $s = rn(r-1, k)$, and let t be large enough. Choose an element b_i among B_i among B_1, \dots, B_s . Consider symmetric exchanges between bases B_1, \dots, B_s and bases B_{s+1}, \dots, B_{s+t} . Let $b_{j,i}$ be an element of B_j for $j = s+1, \dots, s+t$ that exchanges symmetrically with $b_i \in B_i$ for $i = 1, \dots, s$.

Label elements of each B_j among B_{s+1}, \dots, B_{s+t} with numbers $1, \dots, r$. Then each B_j gets a label $(b_{j,1}, \dots, b_{j,s}) \in \{1, \dots, r\}^s$. Since t is large enough we can assume (by choosing a subset and renumbering) that all B_j 's get the same label. That is, the label of $b_{j,i}$ is the same for all j 's. Label B_i with it. At least $n(r-1, k)$ bases B_i have the same label, without loss of generality $B_1, \dots, B_{n(r-1, k)}$. That is, the label of $b_{j,i}$ is the same for all i 's. We can define $b_j := b_{j,i}$. Now, for every $i = 1, \dots, n(r-1, k)$ and every $j = s+1, \dots, s+t$, $b_i \in B_i$ exchanges symmetrically with $b_j \in B_j$.

Consider matroids $M_j := M/b_j|_{B_1 \cup \dots \cup B_{n(r-1, k)}}$ for $j = s+1, \dots, s+t$. Since there are only finitely many matroids on the ground set $B_1 \cup \dots \cup B_{n(r-1, k)}$, if t is large enough there are at least $n(r-1, k)$ j 's for which M_j is the same, without loss of generality for $j = s+1, \dots, s+n(r-1, k)$.

Now, we make symmetric exchanges between bases B_1, \dots, B_n . We exchange $b_i \in B_i$ with $b_{s+i} \in B_{s+i}$ for every $i = 1, \dots, n(r-1, k)$, and obtain bases B'_1, \dots, B'_n . Each B'_i for $i = 1, \dots, n(r-1, k)$ has a distinguished element b_i (former b_{s+i}) such that $b_i \in B_i$ exchanges symmetrically with $b_{i'} \in B_{i'}$ and matroids $M_i := M/b_i|_{B'_1 \cup \dots \cup B'_{n(r-1, k)} \cup B_{n(r-1, k)}}$ are the same.

Consider $n(r-1, k)$ -matroid M_i of rank equal to $r-1$ with $n(r-1, k)$ disjoint bases $B'_1 \setminus b_1, \dots, B'_{n(r-1, k)} \setminus b_{n(r-1, k)}$ (we use the fact that $b_i \in B_i$ exchanges symmetrically with $b_{i'} \in B_{i'}$). From the inductive assumption follows that there are disjoint bases $B''_1, \dots, B''_{n(r-1, k)}$ obtained from $B'_1 \setminus b_1, \dots, B'_{n(r-1, k)} \setminus b_{n(r-1, k)}$ by symmetric exchanges, such that one can label elements of the union (without loss of generality) $B''_1 \cup \dots \cup B''_{n(r-1, k)}$ with labels $1, \dots, r-1$, each B''_j with distinct labels, such that every set of $r-1$ elements of distinct labels forms a basis in M_i .

Observe that bases $B''_1 \cup b_1, \dots, B''_{n(r-1, k)} \cup b_{n(r-1, k)}$ are obtained from bases $B'_1, \dots, B'_{n(r-1, k)}$ by symmetric exchanges. Moreover, one can label elements of the set $B''_1 \cup b_1 \cup \dots \cup B''_{n(r-1, k)} \cup b_{n(r-1, k)}$ with labels $1, \dots, r$ (we use the former labeling, additionally elements b_i get label r), each basis with distinct labels, such that every set of r elements of distinct labels forms a basis in M . This proves the inductive hypothesis. Moreover, we get a recursive bound $n(r, k) \leq rn(r-1, k) + r^{rn(r-1, k)} 2^{rn(r-1, k)} n(r-1, k)$. \square

PROOF OF THEOREM 8. We have to show that $J_M^d = I_M^d$ for $d \geq f(r)$, for some function f . The inclusion $J_M \subset I_M$ implies that $J_M^d \subset I_M^d$ for every d . To prove the opposite inclusion, let $b \in I_M$ be a binomial of degree d , for d large enough.

By Theorem 5 we have that $b = \sum_{i=1}^l a_i b_i$, where each $b_i \in I_M$ is a binomial of degree $(r+3)!$ and each a_i is a monomial of degree $d - (r+3)!$. We will show that $ab \in J_M$, for every binomial $b \in I_M$ of degree $(r+3)!$ and every monomial a of degree d , for d large enough.

We will use Claim 4 from [11]. It asserts that for every basis B of M , if $b \in I_M$ is a binomial, then $y_B^{\deg_B(b) - \deg(b)} b \in J_M$ (where \deg_B is defined by $\deg_B(y_{B'}) = |B' \setminus B|$). In our case $y_B^{r(r+3)!} b \in J_M$ for every basis B of M .

Suppose that $b = y_{D_1} \cdots y_{D_{(r+3)!}} - y_{D'_1} \cdots y_{D'_{(r+3)!}}$ and $a = y_{B_1} \cdots y_{B_d}$ for d large enough. Without loss of generality we can assume that sets $D := D_1 \cup \cdots \cup D_{(r+3)!}$ and B_1, \dots, B_d are pairwise disjoint (if they are not, we introduce parallel elements).

For any k which can be large enough, we take $d = n(r, k)$, and apply Lemma 17 for the d -matroid $M|_{B_1 \cup \cdots \cup B_d}$ with disjoint bases B_1, \dots, B_d . We get that there are disjoint bases B'_1, \dots, B'_d such that

- (1) $y_{B_1} \cdots y_{B_n} - y_{B'_1} \cdots y_{B'_n} \in J_M$,
- (2) among them there are k bases $B'_{i_1}, \dots, B'_{i_k}$, such that M restricted to their union contains a submatroid isomorphic to the k -th blow up of a basis in which $B'_{i_1}, \dots, B'_{i_k}$ are bases.

By (1) we have $ab - y_{B'_1} \cdots y_{B'_n} b = (y_{B_1} \cdots y_{B_n} - y_{B'_1} \cdots y_{B'_n})b \in J_M$, so $ab \in J_M$ if and only if $y_{B'_1} \cdots y_{B'_n} b \in J_M$. In particular, it is enough to show that $y_{B'_{i_1}} \cdots y_{B'_{i_k}} b \in J_M$.

By (2) we have a labeling of elements of bases B'_{i_j} with labels $1, \dots, r$, each B'_{i_j} with distinct labels, such that every set of r elements of distinct labels forms a basis in M . Let V_1, \dots, V_r be label classes, each of them is of size k . Consider a complete r -uniform r -partite hypergraph H with parts V_1, \dots, V_r . Every edge of H is a basis of M .

Since $M|_{B'_{i_1} \cup \cdots \cup B'_{i_k}}$ contains a submatroid isomorphic to the k -th blow up of a basis B , the idea is to make a ‘blow-down’ and instead of $y_{B'_{i_1}} \cdots y_{B'_{i_k}} b$ consider $y_B^k b$ for which we know that if $k \geq r(r+3)!$ then $y_B^k b \in J_M$. However, before doing that we have to make sure that different copies of B behave the same with respect to D . We will use Ramsey theory. A result of Erdős [6] implies the following lemma (see [5] for possible generalizations).

LEMMA 18. *For every integers r, m and c there exists an integer $k = k(r, m, c)$, such that if H is a c -colored (edges receive one of c colors), r -uniform, r -partite, complete hypergraph of size k (size of each part is k), then one can find in it a monochromatic subhypergraph H' of size m .*

Let us define a coloring of H . For each i -element proper subset $S \subset \{1, \dots, r\}$, for each $(r-i)$ -element subset $T \subset D$ and for an edge E of H let a bit $c_{S,T}(E)$ be 1 if the union of S and elements of edge E in parts $V_i, i \in T$ is a basis, and 0 otherwise. We apply Lemma 18 for H with the above coloring ($c \leq 2^{r+r(r+3)!}$) and $m = r(r+3)!$, and get a monochromatic subhypergraph H' of size $r(r+3)!$.

Let $B''_1, \dots, B''_{r(r+3)!}$ be any edges of H' (they are also bases of M) covering vertex set of H' . Let $B''_{r(r+3)!+1}, \dots, B''_k$ be any edges of H completing the covering to the vertex set of H . Notice that $y_{B'_{i_1}} \cdots y_{B'_{i_k}} - y_{B''_1} \cdots y_{B''_k} \in J_M$, since H is a complete hypergraph. Thus it is enough to show that $y_{B''_1} \cdots y_{B''_{r(r+3)!}} b \in J_M$.

Matroid $M|_{D \cup B_1'' \cup \dots \cup B_{r(r+3)!}''}$ contains a submatroid M' isomorphic to $M|_{D \cup B_1''}$ with B_1'' blown-up $r(r+3)!$ times. Thus, since $y_{B_1''}^{r(r+3)!}b \in J_{M'}$ we get that $y_{B_1''} \cdots y_{B_{r(r+3)!}''}b \in J_M$.

We showed that the function $f(r) := n(r, k(r, r(r+3)!), 2^{r+r(r+3)!}) + (r+3)!$ satisfies the required property (where the functions n and k are taken from Lemmas 17 and 18). \square

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